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COMMENT

On the asymptotic properties of solutions of the Korteweg–de Vries equation†

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Abstract. An error is pointed out in a theorem due to Tanaka on the asymptotic behaviour of solutions of the Korteweg–de Vries equation. The appropriate modification leads to a rigorous derivation of the results of Ablowitz and Kodama. A convenient expression for the phase shifts of solitons is given which exhibits the decoupling of the effects due to the presence of both solitons and dispersive wavetrain.

One of the most difficult problems arising in the theory of the inverse scattering method is the rigorous analysis of the asymptotic behaviour of solutions. In particular, much effort has been directed towards the characterisation of the asymptotic properties of solutions of the Korteweg–de Vries (KdV) equation

$$v_t = -v_{xxx} + 6vv_x. \tag{1}$$

Recently, Ablowitz and Kodama (1982) have derived the expressions for the phase shifts of the solitons produced in the evolution of general initial data of (1). However, the arguments used by these authors are heuristic. The aim of the present paper is to show how the expressions obtained by Ablowitz and Kodama may be deduced rigorously from a result due to Tanaka (1975). Our derivation is based on theorem 1.1 of Tanaka (1975) about the limits $t \to \pm\infty$ of (1). It must be noticed that Tanaka proves only the $t \to +\infty$ assertion of his theorem and states the $t \to -\infty$ assertion without proof. As a consequence of theorem 1.1 Tanaka deduces expressions for the motion of the centres of the solitons which are different from the ones obtained by Ablowitz and Kodama. In this paper we prove that the $t \to -\infty$ part of theorem 1.1 of Tanaka is incorrect. Then we formulate the appropriate modification of his statement, which leads us to the results of Ablowitz and Kodama. We provide also an expression of the phase shift formula which displays the decoupling of the contributions of both the discrete and the continuous parts of the spectrum.

We will use the same definition of the scattering data as Tanaka (1975). In this way, we consider the three functions, $a(\xi)$, $b(\xi)$ and $r(\xi) = b(\xi)a(\xi)^{-1}$ associated with the continuous spectrum, and the coefficients η_j and c_j ($j = 1, \ldots, N$) associated with the discrete spectrum. The potential of the Schrödinger operator is determined by the collection $\{r(\xi), \eta_j, c_j\}$. Alternatively, we will use another set of scattering data given by $\{a(\xi), b(\xi), \eta_j, b_j\}'$, where $b_j = |c_j \dot{a}(\mathrm{i}\eta_j)|^{-1}$ ($\dot{a} \equiv \mathrm{d}a/\mathrm{d}\xi$).

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The result proved by Tanaka (1975) is that given a solution of the KdV equation, then for every $\varepsilon > 0$, it follows that

$$\lim_{t\to+\infty}\left\{\sup_{x>\varepsilon t}|v(t,x)-(Sv)(t,x)|\right\}=0, \tag{2}$$

where Sv stands for the reflectionless part of v. In terms of scattering data, S is defined by

$$S\{r(\xi),\eta_j,c_j\}=\{0,\eta_j,c_j\}. \tag{3}$$

On the other hand, Tanaka (1975) states without proof that

$$\lim_{t\to-\infty}\left\{\sup_{-x>-\varepsilon t}|v(t,x)-(Sv)(t,x)|\right\}=0. \tag{4}$$

However, we are going to provide a simple argument which shows that (4) is incorrect. Given a solution $v(t,x)$ of the KdV equation, then $(\hat{P}v)(t,x)=v(-t,-x)$ is also a solution. Therefore, from (2) we deduce

$$\lim_{t\to-\infty}\left\{\sup_{-x>-\varepsilon t}|v(t,x)-(\hat{P}S\hat{P}v)(t,x)|\right\}=\lim_{t\to+\infty}\left\{\sup_{x>\varepsilon t}|(\hat{P}v)(t,x)-(S\hat{P}v)(t,x)|\right\}=0. \tag{5}$$

Hence, by taking into account that $\hat{P}S\hat{P}=PSP$, where P is the space inversion $(Pv)(t,x)=v(t,-x)$, we have that the correct statement about the $t\to-\infty$ behaviour is

$$\lim_{t\to-\infty}\left\{\sup_{-x>-\varepsilon t}|v(t,x)-(PSPv)(t,x)|\right\}=0. \tag{6}$$

In general $PSP\neq S$. To see this, observe that

$$P\{a(\xi),b(\xi),\eta_j,b_j\}'=\{a(\xi),-b(-\xi),\eta_j,b_j^{-1}\}', \tag{7}$$

$$S\{a(\xi),b(\xi),\eta_j,b_j\}'=\{a_s(\xi),0,\eta_j,b_ja_r(\mathrm{i}\eta_j)\}', \tag{8}$$

$$PSP\{a(\xi),b(\xi),\eta_j,b_j\}'=\{a_s(\xi),0,\eta_j,b_ja_r(\mathrm{i}\eta_j)^{-1}\}', \tag{9}$$

where

$$a_s(\xi)=\prod_j\frac{\xi-\mathrm{i}\eta_j}{\xi+\mathrm{i}\eta_j},\qquad a_r(\xi)=\exp\left(\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\frac{\log(1-|r(k)|^2)}{k-\xi}\right),\qquad \mathrm{Im}\,\xi>0. \tag{10}$$

In an equivalent form (9) may be written as

$$PSP\{r(\xi),\eta_j,c_j\}=\{0,\eta_j,c_j|a_r(\mathrm{i}\eta_j)|^2\}. \tag{11}$$

From (2), (3), (6), (11) and the well known results about pure soliton solutions, we conclude that as $t\to\pm\infty$ the solution $v(t,x)$ produces N solitons whose centres move with trajectories $q_j^{\pm}(t)=q_j^{\pm}+4\eta_j^2t$ where

$$q_j^+=\frac{1}{2\eta_j}\log\left(\frac{c_j}{2\eta_j}\right)+\frac{1}{\eta_j}\sum_{l>j}\log\left|\frac{\eta_j-\eta_l}{\eta_j+\eta_l}\right|, \tag{12}$$

$$q_j^-=\frac{1}{2\eta_j}\log\left(\frac{1}{2\eta_j}c_j|a_r(\mathrm{i}\eta_j)|^2\right)+\frac{1}{\eta_j}\sum_{l<j}\log\left|\frac{\eta_j-\eta_l}{\eta_j+\eta_l}\right|. \tag{13}$$

These expressions are equivalent to those found by Ablowitz and Kodama (1982). From (10), (12) and (13) we deduce the following phase shift formula:

$$q_j^+ - q_j^- = \frac{1}{\eta_j} \sum_{l \neq j} \varepsilon(l-j) \log\left|\frac{\eta_l - \eta_j}{\eta_l + \eta_j}\right| + \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \frac{\log(1-|r(k)|^2}{k^2+\eta_j^2}, \tag{14}$$

where $\varepsilon(l-j)$ denotes the sign of $l-j$. The right side of (14) shows clearly the effects on the soliton motion due to the presence of other solitons and of the dispersive wavetrain. In particular, it must be observed that the integral term in (14) is negative. That is to say, solitons are repelled by the dispersive wavetrain. Finally, we notice that recent results (Scharf and Wreszinski 1983) indicate that solutions of (1) tend uniformly to pure soliton solutions as $t^{-1/3}$.

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